

BSSMATH

Introduction Classic Harmonic

Analysis.

over $\mathbb{T}$: $f \in C'(\mathbb{T})$, $e(o) = e^{2\pi i o}$

to denote the exponential 1-periodic

The Fourier coefficients are

$$a_n = \int_0^1 f(t) e(-nt) dt$$

and we have $f(x) = \sum_n a_n e(nx)$

True for $\int$ piecewise continuous. Then

$$S_N(f) = \sum_{|n| < N} a_n e(n x) \rightarrow \frac{f(x^+) + f(x^-)}{2} \quad \text{if } f \in C^1 \quad a_n = o\left(\frac{1}{n}\right)$$

Example $\{x\} = x - [x] - \frac{1}{2}$

$$\{x\} = -\frac{1}{\pi} \sum_{n \geq 1} \frac{\sin(2\pi n x)}{n} = \frac{i}{2\pi} \sum_{n \neq 0} \frac{e(n x)}{n}$$

over $\mathbb{R}$ $f \in \mathcal{S}(\mathbb{R}) \quad f \in C^\infty(\mathbb{R})$

$$f^{(m)}(x) = o\left(\frac{1}{x^m}\right) \quad \forall m, m$$

$$\wedge: \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$$

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e(-\xi x) dx$$

Example $f(x) = e^{-ax^2} \quad \hat{f}(\xi) = \sqrt{\frac{\pi}{a}} e^{-\frac{\pi^2 \xi^2}{a}} \quad a > 0$

Plancherel $\|f\|_2 = \|\hat{f}\|_2, \quad \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle$

Parseval $\|f\|_2 = \left[\sum |a_n|^2 \right]^{1/2} \quad \left[\sum |a_n|^2 \right]^{1/2}$

and we have Poisson summation formula

Theorem $|f| + |f'| + |f''| = O\left(\frac{1}{1+x^2}\right); f \in C^2(\mathbb{R})$

then

$$\sum f(n) = \sum \hat{f}(n)$$

Riemann-Lebesgue

$$|f(x)| \rightarrow 0 \quad |x| \rightarrow \infty$$

Exponential sums

We understand an exponential sum as

$$S_N(f) = \sum_{n=1}^N e(f(n))$$

sum of arithmetics with phase given by a function, somehow smooth.

By considering $g(n) = f(n+M)$

$$\sum_{n=N+1}^{N+M} e(f(n)) = S_{N,M}(f) = \sum_{n=1}^{N+M} e(g(n)) = S_{N+M}(g)$$

Each term of the sum is a wave and we expect cancellation when the function is smooth enough, so it does not have the same phase to interfere constructively.

For example Fourier series can be understood as linear algebra in $L^2(\pi)$ where $\{e(nx)\}$ is a complete orthonormal system. Then adding two terms we get



$\sqrt{2}$

so we get $\sqrt{N}$

of the number of terms

$$f(x) = \sum_{n=1}^N e(nx)$$

$$\|f(x)\|^2 = \sum_{n=1}^N 1 = N \quad \square$$

This is in general the best cancellation we can expect, if f is linear.

For f quadratic, for example

$$\int_0^1 \left| \sum_{n=1}^N e(\alpha n^2) \right|^2 d\alpha = N$$

So on average $F(\alpha)$ can not be smaller than that. Recall that as the phase increases the transform is smaller, but still is not too small.

In general any improvement from the initial bound is good for applications

Example $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \quad \text{Re}(s) > 1$

is absolutely convergent. The famous problem is on the critical strip $s = \sigma + it$ $0 \leq \sigma \leq 1$. in which,

in principle, the function is not defined.

However take $s > 0$ by

$$Z(s) = \sum_{n \geq 1} \frac{(-1)^n}{n^s} \quad \text{Re } s > 1$$

Leibniz. Now

$$\zeta(s) = \frac{1}{2^s} \sum_{n \geq 1} \frac{1}{n^s} + \sum_{n \text{ odd}} \frac{1}{n^s}$$

$$Z(s) = \frac{1}{2^s} \sum_{n \geq 1} \frac{1}{n^s} - \sum_{n \text{ odd}} \frac{1}{n^s}$$

so $\zeta(s) + Z(s) = \frac{1}{2^{s-1}} \zeta(s)$

$\zeta(s) = \left(1 - \frac{1}{2^{s-1}}\right)^{-1} Z(s)$ gives an analytic continuation to $\operatorname{Re}(s) > 0$.

We want to find a relationship between $\zeta(s)$ and an exponential sum

$$\sum_{n=1}^N \frac{1}{n^s} = \sum_{n \leq N} \frac{n^{-it}}{n^{\sigma}}$$

For that we need to remove n^{σ} , and make incomplete the sum.

Lemma (Abel)

$$S_n = \sum_{n=1}^n a_n$$

$$\sum_{n=1}^N a_n f(n) = f(N)S_N - \int_1^N S_t f'(t) dt$$

Proof:

Tomando $q_n = 1$

$$[t] = t - \frac{1}{2}$$

$$\begin{aligned} \sum_{n=1}^N f(n) &= N f(N) - \int_1^N [t] f'(t) dt \\ &= N f(N) - \int_1^N t f'(t) dt + \frac{1}{2} \int_1^N f'(t) dt + \int_1^N \frac{1}{2} t f'(t) dt \\ &= N f(N) - t f(t) \Big|_1^N + \int_1^N t f'(t) dt + \frac{1}{2} \int_1^N f'(t) dt + \int_1^N \frac{1}{2} t f'(t) dt \\ &= \frac{1}{2} (f(N) + f(1)) + \int_1^N f(t) dt + \int_1^N \frac{1}{2} t f'(t) dt \end{aligned}$$

Euler MacLaurin $w=1$. Tomando $f(n) = g(N+n-1)$

queda

$$\sum_{n=N}^M f(n) = \frac{1}{2} (f(N) + f(M)) + \int_N^M f(t) dt + \int_N^M \frac{1}{2} t f'(t) dt$$

Con $f(n) = \frac{1}{n^s}$ queda

$$\sum_{n=1}^N \frac{1}{n^s} = \frac{1}{2} \left(1 + \frac{1}{N^s} \right) + \int_1^N \frac{1}{t^s} dt - s \int_1^N \frac{\frac{1}{2} t^{\frac{1}{2}}}{t^{s+1}} dt$$

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s \int_1^{\infty} \frac{\frac{1}{2} t^{\frac{1}{2}}}{t^{s+1}} dt$$

valida en $\text{Re}(s) > 0$.

$s = \sigma + it$

$$-s \int_1^{\infty} \frac{\frac{1}{2} t^{\frac{1}{2}}}{t^{s+1}} dt = -s \int_1^N \frac{\frac{1}{2} t^{\frac{1}{2}}}{t^{s+1}} dt + O(N^{-\sigma})$$

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} - s \int_1^N \frac{\frac{1}{2} t^{\frac{1}{2}}}{t^{s+1}} dt + O(N^{-\sigma})$$

$$= \frac{1}{2} + \frac{1}{s-1} + \sum_{n=1}^N \frac{1}{n^s} - \frac{1}{2} - \frac{1}{2N^s} - \int_1^N \frac{1}{t^s} dt + O(N^{-\sigma})$$

$$= \sum_{n=1}^N \frac{1}{n^s} + \frac{N^{\frac{1-s}{2}}}{1-s} + O(N^{-\sigma}) \quad \left| \begin{array}{l} \frac{1}{2} \leq \sigma \\ 1 \leq t \leq N \end{array} \right|$$

So to evaluate $\zeta(s)$ at $\sigma = \frac{1}{2}$ or $\sigma = 1$
 we need to evaluate

$$\sum_{n=1}^N n^{-it} = \sum_{n=1}^N e^{-it \log n}$$

By induction we can get (Exercise)

$$\begin{aligned} \sum_{n=N}^M f(n) &= \int_N^M f(t) dt + \frac{1}{2} (f(N) + f(M)) \\ &+ \sum_{n=2}^N \frac{1}{n!} \left(\int_N^M \left(\int_N^{n-1} f(t) dt - \int_N^{n-1} f(t) dt \right) \right) + \int_N^M \int_N^{n-1} f(t) B_n(x - [x]) dx \end{aligned}$$

$$N < M, N \in \mathbb{Z} \quad n \geq 1, f \in C^N([N, M])$$

$$B'_m(x) = m B_{m-1}(x) \quad ; \quad b_n = B_n(0)$$

polynomials of Bernoulli.

$$B_n(x) = -\frac{n!}{(2\pi i)^n} \sum_{n \neq 0} \frac{e(nx)}{n^n}$$

$$b_{2m,n} = 0; \quad S_{2m} = \frac{(2m)!}{(2\pi)^{2m}} \sum_{n \geq 1} \frac{1}{n^{2m}}$$

$\zeta(2m)$ is irrational.



Weyl method

$$f(x) = \alpha_w x^w + \dots + \alpha_0 \in \mathbb{R}[x]$$

$S_n(f)$ is a Weyl sum of degree w .

Let us start with $k=1$.

$$\sum_{n=1}^N e(\alpha n) = \frac{e(\alpha(N+1)) - e(\alpha)}{e(\alpha) - 1}$$

$$e(A) - e(B) = e(C) (e(D) - e(-D))$$

$$C+D = N+1 \quad C = \frac{N}{2} + 1$$

$$C-D = 1 \quad D = \frac{N}{2}$$

$$\begin{aligned} C+D &= 1 \\ C-D &= 0 \end{aligned}$$

$$\frac{2i e\left(\alpha\left(\frac{N}{2}+1\right)\right)}{2i e\left(\frac{\alpha}{2}\right)} \frac{\text{Sen}(\pi N \alpha)}{\text{Sen}(\pi \alpha)}$$

$$|S_N(\alpha)| \leq \frac{1}{|\text{Sen}(\pi \alpha)|}$$

$\alpha = n \pm \frac{1}{N}$
para algún entero n

$$\text{luego } |f(\alpha)| = |f(\|\alpha\|)|$$

Ejercicio: $f(\pi x) \geq 2x \quad \forall x \leq \frac{1}{2}$

Proof: $f(x) = f(\pi x) - 2x$

$$f'(x) = \pi \cos(\pi x) - 2$$

$$f''(x) = -\pi^2 \sin(\pi x) \leq 0 \quad \text{en } 0 \leq x \leq \frac{1}{2}$$

luego f es convexa $f(0) = 0 = f(\frac{1}{2})$

por tanto $f \geq 0$.



$$|S_N(\alpha)| \leq \min\left(N, \frac{1}{2\|\alpha\|}\right)$$

Las sumas exponenciales, dependen algo de la anticomutatividad. Es mucho más fuerte $h=2$. La idea de Weyl es reducir el grado del polinomio $p(u+m) - p(u)$ es un polinomio de grado ≤ 1 menos.

Vamos a hacer un caso particular

$$S_N = \sum_{n=1}^N e\left(\frac{n^2}{N}\right)$$

Lemma $f \in C^2([N, M])$ (Se usa en Van der Corput)

$$\sum_{n=N}^M f(n) = \frac{1}{2} (f(N) + f(M)) + \sum_n \int_N^M f(t) e(n t) dt$$

Proof: Integrate por partes en la fórmula de Euler Mc-Laurin $w=1$, y usar que

$$2 \cos(2\pi n t) = e(ut) + e(-ut)$$

$$S_N = \sum_{-\infty}^{\infty} \int_0^N e\left(\frac{t^2}{N} + ut\right) dt$$

$$t = N\left(x - \frac{u}{2}\right) = Nx - \frac{Nu}{2}$$

$$t^2 = Nx^2 + \frac{Nu^2}{4} - Nu x$$

$$\frac{t^2}{N} = Nx^2 + \frac{Nu^2}{4} - Nu x$$

$$= N \int_{-\infty}^{\infty} e\left(-\frac{Nu^2}{4}\right) e(Nx^2) dx$$

$$\frac{t^2}{N} + ut = Nx^2 + \frac{Nu^2}{4} - Nu x + uNx - \frac{Nu^2}{2}$$

$$= N \int_{-\infty}^{\infty} e(Nx^2) dx + (-i)^{\frac{1}{2}} \int_{-\infty}^{\infty} e(Nx^2) dx = N(1+i^{-1}) I$$

$$= N^{\frac{1}{2}} \frac{1+i^{-1}}{1-i}$$

Por el teorema de los residuos, ⁴¹ se sigue
 (se puede aplicar formalmente

$$f(x) = e(x^2 N) \quad \hat{f}(0) = \sqrt{\frac{1}{-2\pi i N}}$$

$$\sqrt{\frac{1}{-2i}} = \frac{1}{1-i}$$

En general, la idea de Weyl
 es que $p(u+m) - p(u)$ es de un grado
 menor que p .

$$|S_N(\theta)|^2 = \sum_{l, n=1}^N e(\alpha n^2 + \beta n - \alpha l^2 - \beta l)$$

$$= \sum_{n=1}^N e(\alpha(n-l)(n+l) + \beta(n-l))$$

$$= \sum_{n=1}^N e(\alpha(n-l)^2 + 2\alpha l(n-l) + \beta(n-l))$$

$$= \sum_{\substack{n=1 \\ l=1}}^N e(\alpha(n-l)^2 + \beta(n-l)) e(2l\alpha(n-l))$$

$$= \sum_{|w| < N} e(\alpha w^2 + \beta w) \sum_{1 \leq l \leq N-w} e(2l w \alpha)$$

$$w=0$$



$$\ll N +$$

$$1 \leq l + w = n \leq N$$

$$\geq 1$$

$$\left(\sum_{|w| < N} \left| \sum_{1 \leq l \leq N-w} e(2l w \alpha) \right| \right)$$

$$\ll N + \sum_{|w| < N} \min(N-w, \frac{1}{2 \|2w\alpha\|}) \ll N + \sum_{1 \leq w \leq N} \min(N, \frac{1}{\|2w\alpha\|})$$

Si $\alpha = 0$ N^2 ; $\alpha = \frac{1}{2}$ N^2 ; $\alpha = \frac{1}{3}$ $O(N^2)$
 va a depender de la antmehca de α

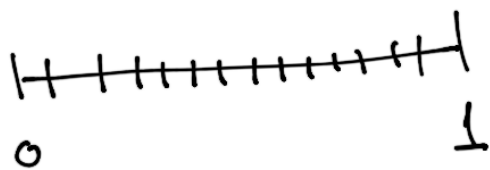
Dirichlet theorem $\alpha, Q \in \mathbb{R}$

$$\exists 1 \leq a < q \leq Q \text{ s.t.}$$

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{1}{qQ} \leq \frac{1}{q^2}$$

$$\frac{h}{Q}, \frac{h+1}{Q}$$

Proof:



$$|q\alpha - a| \leq \frac{1}{Q}$$

For each q take
 $0 \leq q\alpha - [q\alpha] < 1$

pigeonhole principle.

$$\sum_{w \leq X} \min \{N, \|xw\|^{-1}\}$$

$$|\alpha - \frac{q}{f}| < \frac{1}{f^2}$$

$$\frac{XN}{f} + \sum_{l=0}^{\frac{X}{f}} \sum_{j=1}^{f-1} \frac{1}{\frac{j}{f}} = \left(\frac{X}{f} + 1\right) f \lg f$$

$$\frac{N}{f} + N^{\frac{w-1}{f}} \lg f + \lg f$$

$$\sum_{w \leq X} \min \left\{ \frac{N}{K}, \frac{1}{\|xw\|} \right\} \ll \left(\frac{N}{f} + X + f \right) \lg(Xf)$$

By the triangle inequality, when $q \nmid w$,

$$\|2\alpha_k\| \geq \frac{1}{2} \left\| \frac{a}{q} w \right\|$$

$$\sum_{1 \leq w \leq N} \frac{1}{\left\| \frac{a}{q} w \right\|} \leq \frac{N}{q} \sum_{l=0}^{\frac{N}{q}-1} \sum_{lq+1 \leq j \leq (l+1)q-1} \frac{1}{\left\| \frac{a}{q} j \right\|}$$

ka recorre todos los clases mod q

$$= \frac{N}{q} + \sum_{l=0}^{\frac{N}{q}-1} \sum_{j=1}^{q-1} \frac{1}{j} \ll \frac{N}{q} + \left(\frac{N}{q} + 1 \right) q \lg q$$

$$= \frac{N^2}{q} + N \lg q + q \lg q + N$$

$$= \frac{N^2}{q} + N \lg q + q \lg q$$

$$|S_f N| \ll \frac{N}{\sqrt{q}} + \sqrt{N \lg q} + \sqrt{q \lg q}$$

(sumar hasta X)

$$w \geq 2$$

Theorem: Let $P(x) = \sum_{j=0}^w \alpha_j x^j$

$$\left| \alpha_k - \frac{a}{q} \right| \leq \frac{1}{q^2} \quad (a, q) = 1$$

Then
$$\sum_{n=1}^N e(P(n)) \ll N^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{N} + \frac{q}{N^w} \right)^\delta$$

$$\delta = 2^{1-w}$$

Proof:

Lemma $w \geq 1$ $P(x) = \alpha x^w + \dots \in \mathbb{R}(x)$, $\deg P = w$.

$$|S_p(N)| \ll N \left\{ N^{-w} \sum_{|l_i| \leq N} \min \{ N, |k w! (l_1 \cdots l_w)| \}^{-1} \right\} \left(2^{1-w} \right)$$

$$P(u+l) - P(u) = Q(u) \in \mathbb{R}(x) \quad \deg Q = w$$

$$Q(x) = \alpha(u) x^w + \dots$$

$$|S_p(N)|^2 \leq \sum_{|l| < N} \left| \sum_{\substack{1 \leq u \leq N \\ 1 \leq l+u \leq N}} e(Q(u)) \right|$$

$N^{1-w} \delta_{\text{exp}(h, \dots)}$
 $N^{2^{r-1}-w}$

$$\ll N \sum_{|l| < N} \left\{ N^{-w} \sum_{|l_k| < N} \min \left\{ N, \|\alpha(u) l_1 \dots l_{w-1} l\|^{-1} \right\}^2 \right\}^{1-K}$$

By Holder's $p = 2^{1-w} \quad \frac{1}{p} + \frac{1}{q} = 1$

$$1 - \frac{w}{p} + \frac{1}{q} = 2 - \frac{w+1}{p} \quad \square$$

Proof of theorem $w! l_1 \dots l_{w-1} = l < 1^2$

$$\begin{aligned}
 & N^{1-w\delta+\varepsilon} \left(\min \left\{ N, \| \alpha \|^{-1} \right\} \right)^\varepsilon \\
 & \quad \left(|f| < w! \cdot N^{w-1} \right) \\
 & \ll N^{1-w\delta+\varepsilon} \left(\frac{N}{q} + N^{k-1} \lg q + q \lg q \right)^\varepsilon \\
 & = N^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{N} + \frac{q}{N^k} \right)^\varepsilon \quad \square
 \end{aligned}$$

Si queremos pasar a funciones suaves,
 se pueden aproximar por polinomios y
 usar Weyl. Nosotr@s vamos a
 introducir el método de Van der Corput

Una integral oscilatoria, que no es una transformada de Fourier pero es lineal, se comporta como ella, Riemann-Lebesgue e integración por partes

$$I(\lambda) = \int_{-\infty}^{\infty} A(x) e(i\lambda F(x)) dx$$

como varía el resultado, al variar la oscilación.

Sin embargo, si no hay oscilación es como constante y de el término principal.

Theorem: $f \in C^n$ $|f^{(k)}| \in L^1$ $0 \leq k \leq n$

$$|\hat{f}(\xi)| = o(|\xi|^{-n})$$

Proof: Integration by parts $\square$

Theorem: $A \in C^n$ $A^{(k)} \in L^1$ $0 \leq k \leq n$

$$\phi(x) = \frac{1}{F'(x)} \quad \phi^{(k)} \text{ bounded on } \text{supp}(A)$$

then $I(\lambda) = o(|\lambda|^{-n})$

$$I(\lambda) = \int A \phi F' e(F) = A \phi e(F) \Big|_{-\infty}^{\infty} - \int (A \phi' e(F))$$

$$\int |A| < \infty \quad \Rightarrow \quad \lim_{\lambda \rightarrow \infty} A(\lambda) = 0 \quad L^1 \text{ and continuous}$$

$(AS)'$ are in the hypothesis, so apply induction

Now let us check $F'(x)=0$. For

example

$$\int_{-\infty}^{\infty} e^{-x^2} e(-\lambda x^2) dx = \frac{\sqrt{\pi}}{(1+4\pi^2\lambda^2)^{1/4}} = O(|\lambda|^{-1/4})$$

This is the most general case. If

$f'(0)=0$, $f''(0) \neq 0$ then by Taylor

$F(x) \sim \frac{f''(0)}{2} x^2$ close to 0.

Theorem: $A \in C_0^\infty$, $F \in C^\infty$ $F'(0) > 0$

$F'(0) = 0$ is the unique critical point on the $\text{supp}(A)$, then

$$I(\lambda) = \frac{e(\lambda F(0) + \frac{1}{8})}{\sqrt{F''(0) \lambda}} \left(A(0) + a_i \lambda^{-i} + o(|\lambda|^{-n}) \right)$$

Ex $F = x^2$, then

$$I(\lambda) = \frac{1+i}{2\sqrt{\lambda}} \left(\sum_{j=0}^{n-1} \frac{A^{(2j)}(0)}{j! (-8\pi i)^j} \lambda^j + o(|\lambda|^{-n}) \right)$$

Parseval, Taylor, la fórmula de la derivadas con $\hat{A}(z)$, y la fórmula de

Inversión. Si $F(x)$ como en el
Teorema es un cambio de variable

Cuando no es todo $\mathbb{R}$, es una
función característica y se pierde
regularidad

Proposición (Lemas de Van der Corput)

$$F \in C^k([a, b]) \quad F^{(k)}(x) \geq \lambda_k > 0 \quad \forall x \in [a, b]$$

$$\left| \int_a^b e(F(x)) dx \right| \leq C_k \lambda_k^{-\frac{1}{k}}$$

(la oscilación este en el intervalo)

$$w=1$$

$$\left| \int_a^b e(F(x)) dx \right| \leq \frac{1}{\lambda_1}$$

$$F(x) = \lambda_1 \frac{F(x)}{\lambda_1} = \lambda_1 G(x)$$

Proof:

$$\int_a^b \frac{F'}{F'} e(F(x)) = \frac{e(F(x))}{F'} \Big|_a^b - \int_a^b \left(\frac{1}{F'} \right)' e(F(x))$$

$$\leq \frac{1}{F'(b)} + \frac{1}{F'(a)} + \frac{1}{F'(b)} + \frac{1}{F'(a)} \leq \frac{1}{\lambda_1}$$

Q

(I) res to induction making that

$$F^h(x) > 0 \Rightarrow F^h \text{ is monotone } \gamma F^{h-1} \leq \delta \Rightarrow$$

$$\lambda_1^{-1} (F^h(x) - F^h(b)) = |I| \leq 2\delta \lambda_1^{-1}$$

$$\left| \int_h^{\infty} e(P(x)) dx \right| \leq 2\delta \lambda_h^{-1} + C \delta^{-1/h-1}$$

$$\delta \lambda_h^{-1} = \delta^{-\frac{1}{h-1}}$$

$$\delta = \lambda_h^{\frac{h-1}{h}} \quad \boxed{\times}$$

Example: $\int_{N_1}^{N_2} e(x^2 - \lambda x) dx \ll \frac{1}{N_1}$

So it is convergent.

Now the idea is to control exponential sums by oscillatory integrals. This is
 on in the case of monodromy functions

and quite interesting is also the case
when the function oscillates. The first
idea is to use Poisson

$$\sum f(n) = \sum \hat{f}(n)$$

for big n , the terms will be controlled
by Riemann-Liouville type bounds. and
for n small we use Van der Corput or
stationary phase. This is $\sum \varphi(n) f(n)$

For the case in which the function
is not smooth

Proposition: $f \in C^2[a, \beta]$ f' monotone $\alpha < \beta' < \beta \in \mathbb{Z}$

$$\sum_{n=\alpha}^{\beta} e(f(n)) = \sum_{\alpha < u < \beta} \int_u^{\beta} e(f(x) - ux) dx + O(p_g(\beta - \alpha + 1))$$

We can switch $f(n)$ by $f(n) - un$
 so $\alpha = 0$. Now Euler MacLaurin gives

$$\sum_{n=\alpha}^{\beta} e(f(n)) = \int_{\alpha}^{\beta} e(f(t)) dt + \sum_{n \neq 0} \frac{1}{n} \int_{\alpha}^{\beta} f'(x) e(g(x)) dx + O(1)$$

$\int_{\alpha}^{\beta} \frac{f'(x)}{g'(x)} g'(x) e(g(x)) dx \ll \frac{1}{n-p} \int_{\alpha}^{\beta} \left(\frac{f'}{g'} \right)' dx \ll \frac{1}{n-p} \ll p_g(\beta - \alpha)$

So we get

$$\sum e(f(n)) = \int_a^b e(f(x)) + \sum_{0 < n \leq p} \frac{1}{n} \int_a^b f'(x) e(g(x)/n) dx$$

to pp. 21

$$\int_a^b f'(x) e(g(x)) = e(g(b)) - e(g(a))$$

||

$$\sum_{n=1}^b e(g(x))$$

Concluding: $(f' \leq \frac{1}{2})$

$$\sum_{n=1}^b e f(n) = \int_a^b e f(t) dt + O(1)$$

on $[a, b]$

Theorem: $f \in C^2[a, b]$ $0 < \lambda \ll \|f''\| \ll \lambda$

$$\sum_{0 \leq u \leq b} e(f(u)) \ll (b-a) \lambda^{1/2} + O\left(\frac{1}{\lambda^{1/2}}\right)$$

$$\beta - \alpha = \left| f'(x) - f'(y) \right| \approx (b-a) f''(\xi)$$

mean value theorem

$$\beta - \alpha \ll (b-a) \lambda + 1$$

so

$$\sum e(f(u)) = \sum \int_a^b e(f(x) - ux) dx + O((b-a) \lambda^{1/2})$$

$$= \lambda^{1/2} (b-a) + \lambda^{-1/2}$$

if $\lambda < (b-a)^{-2}$ or $\lambda > 1$ it does not

Say anything. If it oscillate too much
we apply Weyl shift.

Theorem $f \in C^W([a, b])$ $W \geq 2$

$$\lambda \leq |f'| \leq h\lambda$$

$$\sum_{0 \leq n \leq b} e(f(n)) \ll h^{\frac{2}{W}} (b-a) \lambda^{\frac{1}{2W-2}} (b-a)^{1-\frac{2}{W}} \lambda^{-\frac{1}{2W-2}}$$

$$W \geq 1^{W-1}$$

Lemma

$$f: [a, b] \rightarrow \mathbb{R}; H \leq b-a$$

$$\left| \sum_{a \leq n \leq b} e(f(n)) \right|^2 \leq \frac{4(b-a)^2}{H} + \frac{4(b-a)}{H} \sum_{1 \leq r < H} \left| \sum_{a \leq n \leq b-r} e(f(n+r) - e(f(n))) \right|$$

$$|S| = \frac{1}{H} \sum_n \sum_{m \in I_n} e(f(n+m))$$

$I_n = [1, H] \cap [a-n, b-n]$. The intersection is non empty when

$$a-n \leq H$$

$$b-n \geq 1$$

$\Rightarrow a-H \leq n \leq b-1$. And Cauchy-Schwarz

$$|S|^2 \leq \frac{(b-a+H)}{H^2} \sum_{a-H \leq n \leq b-1} \left| \sum_{\substack{m \in I_n \\ l \in I_n}} e(f(m+n) - f(l+n)) \right|$$

$$m=l$$

For each y, n there is one l

$$= \frac{(b-a+H)}{H^2} \cdot (b-a+H) \cdot H +$$

$$+ 2 \sum_{a-H \leq n \leq b-1} \sum_{1 \leq r < H} \left| \sum_{\substack{m-l=r \\ l, m \in [1, H] \\ l \neq m}} e(f(m+n) - f(l+n)) \right|$$

$$\sum_{1 \leq r < n} \left| \sum_{a \leq u \leq b-r} e(f(u+nr) - f(u)) \right|$$

$$m+n = m-l + n+l \leq b+r$$

$\lambda \geq 1$ otherwise is trivial $w=2$ as w is even
 hence $w \geq 2$.

$$\sum_{a \leq u \leq b-r} e(f(u+nr) - f(u)) <$$

$$\left| \int_{a-r}^{b-r} (f(u+nr) - f(u)) du \right| = \left| \int_{a-r}^{b-r} (nr) f'(u) du \right| < nr\lambda$$

$$< n^{\frac{2}{w}} (b-a-r) (r\lambda)^{\frac{1}{2w-2}} + (b-a)^{1-\frac{2}{w}} (r\lambda)^{-\frac{1}{2w-2}}$$

$$H = \left[\lambda^{-\frac{1}{w-1}} \right]$$

$$\sum_{j=0}^{\frac{1}{2}Lst} \sum_{N \leq n \leq 2N} e\left(\frac{t}{2\pi} g n\right)$$

$$f(x) = \frac{t}{2\pi} g x; \quad f'(x) = \frac{t}{2\pi} \quad f''(x) = -\frac{t}{2\pi x^2}$$

$$f'''(x) = \frac{t}{\pi x^3} \quad |f''''(x)| \leq \frac{t}{\pi N^3}$$

$$\geq \frac{1}{8} \frac{t}{\pi N^3}$$

$$\lambda = \frac{1}{8} \frac{t}{\pi N^3}$$

$$\log en \text{ base } 2 \quad \text{sub } \frac{5}{12}.$$

Large sieve

Partial summation has been good to treat $\int(\sigma)$. However sometimes

$\sum A(n) e(f(n))$ cannot be small

$A(n) = e(-f(n))$. But, on average, they cannot be proportional to everyone

Proposition: Let $\|x_r - x_v\| > \delta$ then

$$\sum_r \left| \sum_{n=1}^N a_n e(n x_r) \right|^2 \leq C(N + \delta^{-1}) \sum_{n=1}^N |a_n|^2$$

Proof: Take $f, f' \in L^1(\mathbb{T})$

$$\int (x) - \int (y) = \int_I f' dx \quad \int (y) = \min_I \int (x)$$

$$|\int (x)| = \int (y) + \int_I f' dx$$

$$\leq \frac{1}{|I|} \int_I |f| dx + \int_I |f'| dx$$

$$\int (x) = \left(\sum_{n=1}^N a_n e(x u) \right)^2 \quad \text{on } [I_h] < \delta$$

$$\sum_n \left| \sum_{u=1}^N a_u e(u x_j) \right|^2 \ll \sum_n \delta^{-1} \int_{I_n} |g|^2 + \int_{I_n} |g g'|$$

$$\ll \frac{1}{\delta} \int_I |g|^2 + \int_I g g'$$

$$\leq \frac{1}{8} \|g\|^2 + \|g\| \|g'\| \leq \left(\frac{1}{8} + N\right) \tau |a_n|^2$$

Arithmetic large sieve

$$\mu(n) = \begin{cases} (-1)^k & n = p_1 \cdots p_k \\ 1 & n=1 \\ 0 & \text{o.w.} \end{cases}$$

Theorem: $z \in [M+1, M+N]$

$$|z \bmod p| = p - \omega(p) \quad \forall p \leq Q$$

$$|z| \leq \frac{N + Q^2}{\sum_{p \leq Q} h(p)}$$

$$h(p) = \mu^2(p) \prod_{q|p} \frac{\omega(q)}{p - \omega(q)}$$

Bad distribution mod q , increases the
Furber transform mod q if we lift it

we make $\|f\|$ large and by Plancherel $\|f\|$ should be large. So the size of the character χ function is related with the distribution mod p (which is seen on exponential sums as well)

$$\text{Step 1} \quad \chi(q) |Z|^2 \leq \sum_{(q,q)=1} \left| \sum_{n \in Z} e\left(\frac{qn}{q}\right) \right|^2$$

$$\forall \mu(q) \neq 0.$$

$$\left| \sum_{n \in Z} e\left(\frac{bn}{q}\right) \right|^2 = \left| \sum_{j=1}^{p-\omega(p)} \sum_{n \equiv b_j \pmod{p}} e\left(\frac{bn}{q}\right) \right|^2 \leq$$

$$(p - \omega(p)) \sum_{j=1}^p \left| \sum_{n \equiv j \pmod{p}} e\left(\frac{bn}{q}\right) \right|^2$$

$$(p - \omega(p)) \sum_{u \in \mathbb{Z}} \prod_{u \in \mathbb{Z}} \frac{1}{p} \sum_{a=1}^p e\left(\frac{a(u-u)}{p}\right) e\left(\frac{b(u-u)}{q}\right)$$

$$= \frac{p - \omega(p)}{p} \sum_{a=1}^p \left| \sum_{u \in \mathbb{Z}} e\left(\frac{a u}{p}\right) e\left(\frac{b u}{q}\right) \right|^2$$

$$\frac{\omega(p)}{p} \left| \sum_{u \in \mathbb{Z}} e\left(\frac{b u}{q}\right) \right|^2 \leq \sum_{a=1}^{p-1} \left| \sum_{u \in \mathbb{Z}} e\left(\frac{a u}{p}\right) e\left(\frac{b u}{q}\right) \right|^2$$

C.R.T. ends the result is induced.

Theorem $\pi_2(N) \leq C \frac{N}{(\log N)^2}$

$$\mathbb{Z} = \left\{ n \leq N : n \not\equiv 0, -2 \pmod{p} \text{ for } p < \sqrt{N} \right\}$$

$$|\mathbb{Z}| = \pi_2(N) + \sqrt{N} \leq \sum_{q \leq \sqrt{N}} \omega(q)$$

$$\sum_{n \leq N} \frac{1}{n} \sim \prod_{p \leq N} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{p \leq N} \frac{p}{p-1}$$

$$= \prod_{p \leq N} \frac{p^2}{p^2-1} \prod_{p \leq N} \frac{p+1}{p} \leq \sum_{p \leq N} (2) \prod_{p \leq N} \left(1 + \frac{1}{p}\right)$$

$$\left(\sum_{n \leq N} \frac{1}{n}\right)^2 \leq \sum_{p \leq N}^2 \prod_{p \leq N} \left(1 + \frac{2}{p} + \frac{1}{p^2}\right)$$

$$= \sum_{p \leq N}^2 \prod_{p \leq N} \left(1 + \frac{1}{p^2(1 + \frac{2}{p})}\right) \prod_{p \leq N} \left(1 + \frac{2}{p}\right)$$

$$\leq \left(\sum_{p \leq N} (2)\right)^2 \prod_{p \leq N} \left(1 + \frac{1}{p^2}\right) \prod_{p \leq N} \left(1 + \frac{2}{p}\right)$$

$$\leq \sum_{p \leq N}^3 \prod_{p \leq N} \left(1 + \frac{2}{p}\right) \sim \sum_{p \leq N}^3 \sum \frac{2^{\omega(p)}}{p}$$

$$\sum_{q \leq N} \frac{d(q)}{q} = \sum_{d \leq N} \frac{1}{d} \sum_{\substack{q \leq \frac{N}{d}}} 1 \sim \sum_{d \leq \frac{N}{d}} \frac{1}{d} \sim \frac{1}{2} \log \frac{N}{d}$$

$$\sim \frac{1}{2} (\log N)^2 - \frac{(\log N)^2}{2} \sim \frac{(\log N)^2}{2} \quad \text{B}$$

$$\frac{1}{4} \sum_{q \leq \frac{N}{4}} \frac{d(q)}{q} \sim \frac{1}{4} \log \frac{N}{4}$$

$$\sum_{p \leq \frac{1}{2}} \frac{1}{p^2} \text{ small.}$$

$$h(q) = \prod_{p|q} \frac{2}{p-2}$$

$$\sum_{q \leq Q} h(q) = \sum_{q \leq Q} \prod_{p|q} \frac{2}{p-2}$$

$$= \sum_{q \leq Q} \frac{2^{\omega(q)}}{q} \prod_{p|q} \frac{1}{1 - \frac{2}{p}} < \sum_{q \leq Q} \frac{2^{\omega(q)}}{q^2 q}$$

$$\sum_{q \leq Q} \frac{\sum_{d|q} 1}{q^2 q} = \sum_{d < Q} \frac{1}{d^2} \sum_{\substack{q \leq Q \\ d|q}} \frac{1}{q}$$

$$\sim \frac{1}{Q^2} N$$

THEOREM 3. Let P be a polynomial with integral coefficients. For any positive integers N, q there exists an integer h such that

$$\left| \sum_{n=1}^N e\left(\frac{P(n) + hn}{q}\right) \right| \geq \frac{N}{\sqrt{q}}.$$

PROOF. Let $c(r)$ denote the number of integers n , $1 \leq n \leq N$, such that $n \equiv r \pmod{q}$. Thus

$$\sum_{h=1}^q \left| \sum_{n=1}^N e\left(\frac{P(n) + hn}{q}\right) \right|^2 = \sum_{h=1}^q \left| \sum_{r=1}^q c(r) e\left(\frac{P(r) + hr}{q}\right) \right|^2.$$

By the orthogonality relations among the additive characters $\pmod{q}$ we see that this is

$$= q \sum_{r=1}^q |c(r)|^2.$$

By Cauchy's inequality this is

$$\geq \left(\sum_{r=1}^q c(r) \right)^2 = N^2.$$

This gives the stated result.

From Theorem 3 we see that the first term in the bound (10) of Theorem 1 can not be reduced. For Weyl sums of degree $k > 2$ we now exhibit a better lower bound, but only for certain special values of q .

THEOREM 4. Let N, k, p, a be given, with $p > k \geq 2$, $(a, q) = 1$, and p prime. Then there exists a polynomial $P \in \mathbb{R}[x]$ of degree k and leading coefficient a/p^k such that

$$\left| \sum_{n=1}^N e(P(n)) \right| \geq N/p.$$

If we put $q = p^k$, then the leading coefficient is a/q , and the Weyl sum is larger than $Nq^{-1/k}$.

PROOF. We first show that

$$(11) \quad \sum_{n=1}^{p^k} e\left(\frac{an^k}{p^k}\right) = p^{k-1}.$$

To see this, write $n = mp^{k-1} + r$. Then $an^k \equiv akmp^{k-1}r^{k-1} + ar^k \pmod{p^k}$, so the sum is

$$\sum_{r=1}^{p^{k-1}} e\left(\frac{ar^k}{p^k}\right) \sum_{m=1}^p e\left(\frac{akmr^{k-1}}{p^k}\right).$$

Metodo del círculo

Problem: Given a set $C \subseteq \mathbb{Z}$

Estimate $\Gamma(n) = \#\{c_1 + \dots + c_n = n : c_i \in C\}$.

$F(z) = \sum_{c \in C} z^c$ analytic in $|z| < 1$.

$$G(z) = F(z)^n = \sum_{c_1, \dots, c_n \in C} z^{c_1 + \dots + c_n} = \sum_n \Gamma(n) z^n$$

By a change of variable $z = e^{2\pi i \theta}$

$F(\theta) = \sum_c e^{2\pi i \theta c}$ will be bigger in

$\theta = \frac{a}{b}$ b small.

By Cauchy's theorem.

$$\Gamma_{C,h}(w) = \frac{1}{2\pi i} \int_C \frac{G(z)}{z^{h+1}} dz$$

It has prehistory in Weyl's uniform distribution: V_N measure of 1 at a_i

$$\hat{V}_N(w) = \sum_{n=1}^N e(-wv_n)$$

$$Z(N; \alpha) = \text{card } n \in \mathbb{Z} \quad 1 \leq v_n \leq \alpha \quad \Bigg\}$$

$$\lim_{N \rightarrow \infty} \frac{1}{N} Z(N; \alpha) = \alpha$$

v_n uniformly distributed $\Leftrightarrow \hat{V}_N(w) \rightarrow 0$ faster

rather than this we will use

$$F(z) = \sum_{m \leq N^{1/w}} z^{m^w}$$

so Cauchy's formula is still valid for any $n \leq N$, and we avoid convergence problem.

$$F\left(\frac{a}{q} + \beta\right) = \underbrace{\sum_{r=1}^q e\left(\frac{ar^w}{q}\right)}_{S(a, q)} \sum_{\substack{n \leq N^{1/w} \\ n \equiv r \pmod{q}}} e(\beta n^w)$$

β is very small so we control the oscillation, and can approximate it by an integral to get

$$F\left(\frac{a}{q} + \beta\right) \sim \frac{S(a, q)}{q} \int_0^{N^{1/w}} e(\beta y^w) dy$$

$$= \frac{S(a, q)}{q} N^{1/w} \int_0^1 e(\beta N u^w) du = I \frac{S(a, q)}{q} N^{1/w}$$

which is a constant times $N^{1/w}$

$$M_{a, q} = \left\{ \alpha \in [0, 1] : \left| \alpha - \frac{a}{q} \right| \leq N^{-1 + \frac{1}{2w}} \right\}$$

$$q \leq N^{\frac{1}{2w}}$$

$$\int_{M_{a, q}} F^s(x) e(-Nx) dx \sim N^{\frac{s}{w}} \frac{S(a, q)}{q^s} \int_{-N^{-1 + \frac{2}{w}}}^{N^{-1 + \frac{2}{w}}} I^s e(-N(\beta r \frac{a}{q})/4)$$

$$\sim N^{\frac{s}{w} - 1} \left(\frac{S(a, q)}{q} \right)^s e(-N \frac{a}{q}) \int_{-N^{1/w}}^{N^{1/w}} \left| \int_0^1 e(tu^w) du \right|^s e(-t) dt$$

By Van der Corput

$$\int_0^1 e(tu^w) dt < t^{-1/w}$$

so the integr. is abs. conv. for $s > w$.

$$\text{So } \int_{m(a,q)} f^s e(-Nx) dx \sim e N^{\frac{s}{w}-1} \frac{\sum_{(a,q)=1}^s e(\frac{a}{q})}{q^s} e(N \frac{a}{q})$$

$$\sum_{q \leq N^{\frac{1}{2w}}} \sum_{(a,q)=1}^s \frac{e(\frac{a}{q})}{q^s} e(-N \frac{a}{q})$$

$$\int_m \sim O(N) e N^{\frac{s}{w}-1} + o(N^{\frac{s}{w}-1})$$

by Weyl method

$$S(a, q) = \left| \sum_{r=1}^q e\left(\frac{ar^w}{q}\right) \right| < q^{1-\delta}$$

and so

$$G(N) = \sum_{q \geq 1} \sum_{\substack{(a, q) = 1 \\ (a, q) = 1}} \frac{S(a, q)}{q^s} e\left(-N \frac{a}{q}\right)$$

$$|G(N)| < \sum_{q \geq 1} \frac{1}{q^{8s-1}} < \infty \quad s > \frac{2}{8}$$

$$S = 2^w + 1$$

For minor arcs we need a lemma

Lemma (Hua) $1 \leq j \leq K$

$$\int_0^1 \left| \int(\alpha) \right|^{2^j} d\alpha \ll N^{2^{-j} + \varepsilon}$$

$$\left| \int(\alpha) \right|^{2^j} = N^{2^{-j}-1} \sum_{h_j} e(\alpha h_1 \dots h_j f_j(\alpha))$$

$$\ll N^{2^{-j}-1} \sum_h c_h e(\alpha h)$$

$$c_0 < N^j \quad c_h < N^\varepsilon \quad \forall h \neq 0$$

$$\left| \int(\alpha) \right|^{2^j} = \sum_h b_h e(\alpha h)$$

$$b_h = \left\{ x_1^h + \dots + x_{j-1}^h - y_1^h + \dots - y_{j-1}^h = h \right\}$$

$$b_0 = N^{2^{j-1} + \epsilon}$$

$$\sum b_h = N^{2^j}$$

$$\int_0^1 |f(\omega)|^{2^j}$$

$$\int_0^1 |f(\omega)|^{2^{j+1}} d\omega = \int_0^1 |f(\omega)|^{2^j} |f(\omega)|^{2^j} < N^{2^{j-1}-1} \sum b_h$$

$$Z < N^{2^{j-1}-1} (N^{2^j + \epsilon}) = N^{2^{j+1} - j - 1 + \epsilon} \quad \square$$

Minim $W(s)$

$\alpha \in m \quad 7 > N^2$

$$f(\omega) < N^{1 - \frac{\nu}{4} + \epsilon} = 1$$

$$\int_m |f(\omega)|^s < |f(\omega)|^{5-2^N} \int_0^1 |f(\omega)|^{2^N}$$

$$N^{(5-2^N)(1-\frac{\nu}{4}) + 2^N - h + \epsilon}$$

$$= N^{5 - (5-2^N)\frac{\nu}{4} - 1} \quad \square$$

Circl method

$$C \subset \mathbb{N} \quad \Gamma_{C,h}(u) = \#\{c_1 + \dots + c_h = u\}$$

$$f(z) = \sum_{c \in C} z^c$$

$$f^h(z) = \sum_u \Gamma_{C,h}(u) z^u$$

$$f(\alpha) = \sum_{n \leq N^{1/h}} e(n^h \alpha)$$

$$f^s(\alpha) = \sum_{n \leq N} \Gamma_h(n) e(n\alpha)$$

$$\Gamma_h(N) = \int_0^1 f^s(\alpha) e(-N\alpha) d\alpha$$

$f(\alpha)$ is big if $\alpha \geq \frac{a}{2}$

$$\int_{M(a, q)}$$

$$+ \int_{m(a, q)}$$

$$f\left(\frac{a}{q} + \beta\right) = \sum_{j=1}^q e\left(\frac{j^2 a}{q}\right) \sum_{\substack{m \equiv j(q) \\ m \leq N^{1/w}}} e(m^w \beta)$$

$$\frac{1}{q} \sum_{j=1}^q e\left(\frac{j^2 a}{q}\right) \int_0^{N^{1/w}} e(y^w \beta) dy$$

$$y = N^{1/w} t$$

$$= N^{1/w} \frac{S(a, q)}{q} \int_0^1 e(N \beta t^w) dt$$

$$N^{s/w} \left(\frac{S(a, q)}{q} \right)^s \int_{-N^{-1+\frac{1}{2w}}}^{N^{-1+\frac{1}{2w}}} \left(\int_0^1 e(it\beta) dt \right)^s e(-N\beta) d\beta$$

$$N^{s/w-1} \left(\frac{S(a, q)}{q} \right)^s e(-N\frac{a}{q}) \int_{-N\frac{1}{2w}}^{N\frac{1}{2w}} I_1^s e(-\beta) d\beta$$

$$\left| \int_0^1 e(it\beta) dt \right| \leq \beta^{-s/w} \quad s > w$$

$$\sim O(N^{\frac{s}{w}-1} \left(\frac{S(a, q)}{q} \right)^s e(-N\frac{a}{q}) \frac{1}{q^{\frac{1}{2}(\frac{1}{q} + \frac{1}{q})}})$$

$$\sum_{q \leq N^{\frac{1}{2w}}} \sum_{(a, q)=1} \left(\frac{S(a, q)}{q} \right)^s e(-N\frac{a}{q}) = O(N)$$

$$q^{1-\delta}$$

$$\frac{1}{q^{\delta s}}$$

$$\delta s \geq 1 > 1$$

$$s > \frac{2}{\delta} = 2^w$$

$$15 = 2^w \cdot 11$$

main axes

Hug's Lemma

$$\int_0^1 |f(x)|^{2^w} dx \ll N^{2^w - w + 2}$$

$$|f(x)|^{2^j} \ll N^{2^j - j - 1} \sum c_h e(\alpha h)$$

$$h_1 \dots h_j, P_j(x, h_1, \dots, h_j) = h \quad N^j \quad h \approx$$

$$N^2 \quad h > 0$$

$$|f(x)|^{2^j} = \sum b_h e(\alpha h) \quad S_0 = N^{2^j + j}$$

$$\int |f(x)|^{2^{j+1}} \ll N^{2^{j+1} - j - 1} \sum c_h S_h$$

$$N^{2^j - j + 1 + \epsilon} + N^{2^j}$$

Shift Weyl

$$\left| \sum_{n=a}^b e(f(n)) \right|^2 \leq \frac{4(b-a)^2}{H} + \frac{4(b-a)}{H} \sum_{1 \leq n \leq a} \left| \sum_{a \leq n \leq b-H} e(f(n+g)) \right|$$

$$\left| \sum_{n=a}^b e(f(n)) \right| = \frac{1}{H} \sum_{m=1}^H \sum_{a-m \leq n \leq b-m} e(f(n+m))$$

$$= \frac{1}{H} \sum_{a-H \leq n \leq b-1} \sum_{m \in I_n} e(f(n+m))$$

$$\left| \right|^2 \leq \frac{(b-a+H)}{H^2} \left(\sum_{a-H \leq n \leq b-1} \left| \sum_{m \in I_n} e(f(n+m)) \right| \right)^2$$

$$= \frac{(b-a+H)}{H^2} \left(\sum_n \sum_{m, p \in I_n} e(f(n+m)) - f(n+p) \right)$$

$$= 2 \frac{(b-a+H)}{H^2} \sum_{1 \leq p < m \leq H} \sum_{\substack{a \leq n+m \leq b \\ a \leq n+p \leq b}}$$

$$\frac{b-rN}{N} \sum_{-r \leq m \leq b-r} \tau$$

Theorem $\lambda < (\int^h) < h \lambda$

$$h^{\frac{2}{N}} (b-a) \int^{\frac{1}{2N-2}} + (b-a)^{1-\frac{2}{N}} \int^{-\frac{1}{2N-2}}$$

$$\boxed{H = \lambda^{-\frac{1}{N-1}}}$$

Application large sieve

$$\tau \left| \sum_{n \leq N} a_n e\left(\frac{n}{m}\right) \right|$$

$\{x_v\}$ δ spaced

$$\sum_v \left| \sum_{n=1}^N a_n e(n x_v) \right|^2 \leq C(N+\delta)^{-1} / \tau k_v$$

Arithmetic:

Theorem $Z \subset \{1, N\}$

$$|Z \bmod q| = p^{\omega(q)} \quad \forall p \leq Q.$$

$$|Z| \leq \frac{N + Q^2}{\sum h(q)}$$

$$h(q) = \mu^2(q) / \prod_{p|q} \frac{p^{\omega(p)}}{p^{\omega(p)}}$$

$$\left| \sum_{u \in Z} e\left(\frac{au}{q}\right) \right|^2 = \left| \sum_{j=1}^{p-\omega(p)} \sum_{u \equiv b_j} e\left(\frac{au}{q}\right) \right|^2$$

$$\leq (p-\omega(p)) \sum_{j=1}^p \left| \sum_{u \equiv j(p)} e\left(\frac{au}{q}\right) \right|^2$$

Lemma: $\sum_{j=1}^p \left| \sum_{u \equiv j(p)} \delta_u \right|^2 = \frac{1}{p} \sum_{j=1}^p \left| \sum_u \delta_u e\left(\frac{ju}{p}\right) \right|^2$

$$= (p - \frac{\omega(p)}{p}) \sum_{j=1}^p \left| \sum_u e\left(\frac{au}{q}\right) e\left(\frac{ju}{p}\right) \right|^2 = \frac{p-\omega(p)}{p} \sum_{j=1}^p \left| \sum_u e\left(\frac{au}{q} + \frac{ju}{p}\right) \right|^2$$

Trivial problem

$$\pi_2(N) \ll \frac{N}{\lg^2 N}$$

$$\sum \frac{1}{p} = \infty.$$

$$Z = \{ n \leq N : n \neq 0, \chi(n) \neq 0, \forall p \leq \sqrt{N} \}$$

$$|\pi_2(N)| \leq |Z| + \sqrt{N}.$$

$$\sum h(q) \geq \sum_{q \leq \sqrt{N}} \frac{2^{\omega(q)}}{q}$$

$$\sum_{n \leq N} d(n) = N \lg N + o(N)$$

$$\sum_{n \leq \frac{N}{p^2}} d(np^2) \leq 3 \frac{N}{p^2} \lg N$$

$$p \geq 5 \text{ d.v.}$$

$$p=2, \quad p=3$$

$$\sum_{n \leq \frac{N}{4}} d(4n) = \sum_{j=0}^{\lg_2 N} d(2^{j+2}) \sum_{n \leq \frac{N}{2^{j+2}}} d(n)$$

$$= N \lg N \sum_{j=0}^{\lg_2 N} \frac{j+3}{2^{j+2}} = \frac{1}{2} N \lg N$$

$$\sum_{j=0}^{\infty} \frac{j}{2^{j+2}} = \frac{1}{8} \sum_{k \geq 0} k 2^{-k} = \frac{1}{4}$$

$$\sum_{k \geq 0} k x^k = \frac{1}{1-x^2}$$

$$\sum k x^{k-1} = \frac{1}{(1-x)^2}$$

$$\sum_{j \geq 0} \frac{\lg 3^N}{d(3^N)} \sum_{h \leq \frac{N}{3^{N+2}}} d(h)$$

$$\frac{N \lg N}{27} \sum_{h \geq 0} \frac{h}{3^{N-1}}$$

$$\frac{1}{(1 - \frac{1}{3})^2} \frac{9}{4}$$

Exercise

$$\Pi(M+N) - \Pi(M) \leq \frac{1}{2} \frac{N}{\lg N} + O\left(\frac{N}{\lg N}\right)$$

Circle method

Exercise: Weyl shift
 $\mu \mapsto \mu + \epsilon$

$$N^{2L_j-1} N^{L_j+\epsilon}$$

Bibliography: Ten lectures between
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